On classification of second-order PDEs possessing partner symmetries

2009 J. Phys. A: Math. Theor. 42 395202

(http://iopscience.iop.org/1751-8121/42/39/395202)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 395202 (20pp) doi:10.1088/1751-8113/42/39/395202

On classification of second-order PDEs possessing partner symmetries

M B Sheftel[1,2] **and A A Malykh**[3]

[1] Department of Physics, Boğaziçi University, 34342 Bebek, Istanbul, Turkey
[2] Department of Higher Mathematics, North Western State Technical University, Millionnaya St 5, 191186, St Petersburg, Russia
[3] Department of Numerical Modelling, Russian State Hydrometeorological University, Malookhtinsky pr. 98, 195196 St Petersburg, Russia

E-mail: mikhail.sheftel@boun.edu.tr and andrei-malykh@mail.ru

Received 21 April 2009, in final form 18 August 2009
Published 8 September 2009
Online at stacks.iop.org/JPhysA/42/395202

Abstract

Recently we have demonstrated how to use partner symmetries for obtaining noninvariant solutions of the heavenly equations of Plebański that govern heavenly gravitational metrics. In this paper, we present a classification of scalar second-order PDEs with four variables that possess partner symmetries and contain only second derivatives of the unknown. We present a general form of such a PDE together with recursion relations between partner symmetries. This general PDE is transformed to several simplest canonical forms containing two heavenly equations of Plebański among them and two other nonlinear equations which we call the mixed heavenly equation and asymmetric heavenly equation. We have calculated all the point and contact symmetries of all the canonical equations which can be used as an input in our recursion relations. On an example of the mixed heavenly equation, we show how to use partner symmetries for obtaining noninvariant solutions of PDEs by a lift from invariant solutions. Finally, we present Ricci-flat self-dual metrics governed by solutions of the mixed heavenly equation and its Legendre transform.

PACS numbers: 02.30.Jr, 04.20.Jb
Mathematics Subject Classification: 35Q75, 83C15

1. Introduction

In his paper [1], Plebański introduced heavenly equations for a single potential generating (anti-)self-dual heavenly metrics which satisfy complex vacuum Einstein equations. Two real cross sections of these complex metrics, Kähler metrics with Euclidean or ultra-hyperbolic signature, are generated by the elliptic and hyperbolic complex Monge—Ampère equation

(CMA) respectively, the particular cases of the first heavenly equation. Solutions of CMA play an important role in the theory of gravitational instantons [2], where all gravitational metrics obtained so far, apart from the ones that we obtained lately [3, 4], have Killing vectors, i.e. admit continuous symmetries. This implies symmetry reduction in the number of independent variables in metric components [5], so these metrics actually live on manifolds of dimensions less than 4.

Recently we introduced the concept of partner symmetries and applied them to obtain noninvariant solutions of the complex Monge–Ampère equation and the second heavenly equation of Plebański [3, 4, 6, 7]. Only such solutions could generate heavenly metrics with no Killing vectors, so that the metric components would depend on all four independent variables. Partner symmetries constitute a certain type of nonlocal symmetries and solution, that are invariant with respect to these nonlocal symmetries, are generically noninvariant solution in the usual sense, i.e. they depend on all four variables, so that no symmetry reduction in the number of independent variables occurs. The idea of using the invariance under nonlocal symmetries in order to get noninvariant solutions, suggested first by Dunajski and Mason [8, 9], clarified the meaning of differential constraints, which we used earlier in [10, 11] to derive noninvariant solutions of CMA. Since the partner symmetries and their use for lifting invariant solutions to noninvariant ones [4, 7] proved to be an appropriate tool for constructing noninvariant solutions of partial differential equations (PDEs) and a PDE taken at random would not admit partner symmetries, the natural question arises: how general this method of obtaining noninvariant solutions can be, or, in other words, what is the general form of equations that possess partner symmetries?

To give a partial answer to this question, in this paper we present some results on a classification of the second-order PDEs of the general form

$$F(u_{tt}, u_{tx}, u_{ty}, u_{tz}, u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}, u_t, u_x, u_y, u_z, u, t, x, y, z) = 0 \tag{1.1}$$

that possess partner symmetries. Here u is the unknown that depends on the four independent variables t, x, y, z and the subscripts denote partial derivatives of u, e.g. $u_{tt} = \partial^2 u/\partial t^2$, $u_{tx} = \partial^2 u/\partial t\partial x \dots$. Although we have derived a complete set of equations for F such that equation (1.1) admits partner symmetries, we are currently able to give a general solution to these equations only for F that depends only on second derivatives of u. Thus, we obtain a classification of PDEs of the form

$$F(u_{tt}, u_{tx}, u_{ty}, u_{tz}, u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}) = 0 \tag{1.2}$$

that possess partner symmetries.

Our definition of partner symmetries requires the following two conditions to be satisfied.

(i) The symmetry condition for PDE (1.1) (determining an equation for symmetries) has the form of a two-dimensional divergence that implies the existence of a single potential for each symmetry.

(ii) The potential of each symmetry is itself a symmetry of (1.1), i.e. a partner symmetry for the original symmetry.

The first condition is satisfied in two steps: at first we require the symmetry condition to have the form of a four-dimensional divergence and then reduce this to a two-dimensional divergence by imposing additional constraints on F. We note that it was also possible to use a four-dimensional divergence form by introducing several potentials, as was shown, for example, by Bluman and Kumei [12], which would probably modify our concept of partner symmetries. This work is currently in progress.

In section 2, we derive the restriction on the form of equation (1.1) implied by the requirement that the symmetry condition should have the form of a four-dimensional divergence: the left-hand side of equation (1.1) itself should be a four-dimensional divergence, so that (1.1) becomes a conservation law.

In section 3, we derive further conditions on F under which the four-dimensional divergence form of the symmetry condition is reduced to a two-dimensional divergence form which implies the existence of a single potential for each symmetry of (1.1).

In section 4, we require that the potential of a symmetry should itself be a symmetry of equation (1.1) and obtain the final set of equations for F. The definition of the symmetry potential then becomes a recursion relation for symmetries which generically maps any local symmetry into a certain nonlocal symmetry. To have an explicit form of this recursion relation, we still need a solution of the equations for F. We note that our *symmetry potential* is completely different from *potential symmetries* of Bluman and Kumei [12], where potentials are introduced not for symmetries but for PDEs, set in a divergence form, and symmetries are allowed to depend on these potential variables.

In section 5, we attempt to solve the set of equations for F. The solution process in full generality turns out to be too lengthy and suggests many cases and subcases to be considered. Therefore, here we restrict ourselves to the case where F in (1.1) depends only on the second derivatives of u and the equation takes the form (1.2). Then we obtain a general solution for F on the left-hand side of (1.2), that is, a general form of the second-order PDE with four variables containing only second derivatives of u that possesses partner symmetries, up to a change of notation for independent variables. We also obtain a recursion relation between partner symmetries in an explicit form.

In section 6, we derive a complete set of canonical forms, to which the general PDE with partner symmetries can be transformed by point and Legendre transformations, together with recursions for symmetries of these canonical equations. Among these canonical forms we find the first and second equations of Plebañski and two other nonlinear equations which we call the mixed heavenly equation and asymmetric heavenly equation. The mixed heavenly equation turns out to be related by a partial Legendre transformation to Husain's heavenly equation [13–15], which is an alternative form of the self-dual gravity equation related to the chiral model approach to self-dual gravity.

In section 7, we find all the point and contact symmetries of the canonical equations. These symmetries can be used in the recursion relations both for the original symmetry and for its potential. The recursion relations then become differential constraints that determine solutions which are invariant under a certain nonlocal symmetry closely related to the partner symmetries. However, they are still noninvariant solutions in the usual sense, with no symmetry reduction in the number of independent variables.

In section 8, we demonstrate an application of partner symmetries for finding noninvariant solutions of PDEs on an example of the mixed heavenly equation. We choose both symmetries in the recursion relations as translational symmetries, with the recursions becoming differential constraints, and then show that Legendre transformation in two variables of both the equation and two differential constraints leads to a set of three linear equations with constant coefficients. One of these equations depends only on three variables, containing the fourth variable merely as a parameter, and coincides with the Legendre transform of the translational symmetry reduction of the mixed heavenly equation, but expressed in new variables. Two other linear equations provide a lift of any solution of this equation, which is an invariant solution to the mixed heavenly equation, to a noninvariant solution that depends on all four variables. We present explicitly a linear combination of exponential solutions and a polynomial solution as examples of such solutions.

In section 9, we obtain Ricci-flat *mixed heavenly metric* in the self-dual gravity, governed by solutions of the mixed heavenly equation, by using a one-dimensional Legendre transformation of Husain's heavenly metric with a subsequent symmetrization of the transformed metric. Then we apply the linearizing Legendre transformation from section 8 to the mixed heavenly metric to obtain the Ricci-flat self-dual metric with a potential satisfying the Legendre transformed mixed heavenly equation. We are now able to use our solutions of the latter equation, that are given in section 8, in the obtained metric or any other solutions of the above-mentioned three linear PDEs with constant coefficients. In this way we arrive at an explicit form of a Ricci-flat self-dual metric with components generically depending on all four independent variables which, as a consequence, will admit no continuous symmetries.

We have to mention also that a different classification of integrable three- and four-dimensional PDEs, that contain only second derivatives of the unknown, was given by Ferapontov *et al* in [16, 17]. In this approach integrability is understood as the existence of sufficiently many hydrodynamic reductions, which is a requirement completely different from the existence of partner symmetries and therefore the results are also completely different.

2. Divergence form of symmetry condition

Let φ be a symmetry characteristic [18] of (1.1). Then the symmetry condition for symmetries φ admitted by (1.1) is determined by vanishing of the Fréchet derivative of F on solutions of (1.1):

$$\begin{aligned}\hat{A}(\varphi) \equiv F_u\varphi + F_{u_t}\varphi_t + F_{u_x}\varphi_x + F_{u_y}\varphi_y + F_{u_z}\varphi_z + F_{u_{tt}}\varphi_{tt} + F_{u_{tx}}\varphi_{tx} + F_{u_{ty}}\varphi_{ty} + F_{u_{tz}}\varphi_{tz}\\ + F_{u_{xx}}\varphi_{xx} + F_{u_{xy}}\varphi_{xy} + F_{u_{xz}}\varphi_{xz} + F_{u_{yy}}\varphi_{yy} + F_{u_{yz}}\varphi_{yz} + F_{u_{zz}}\varphi_{zz} = 0,\end{aligned} \tag{2.1}$$

where $\varphi_t = D_t\varphi$, $\varphi_x = D_x\varphi, \ldots$ and $D_t, D_x, \ldots$ denote operators of total derivatives with respect to $t, x, \ldots$, e.g.

$$\begin{aligned}D_t f = \partial f/\partial t + u_t\partial f/\partial u + u_{tt}\partial f/\partial u_t + u_{xt}\partial f/\partial u_x + u_{yt}\partial f/\partial u_y\\ + u_{zt}\partial f/\partial u_z + u_{ttt}\partial f/\partial u_{tt} + u_{txt}\partial f/\partial u_{tx} + u_{tyt}\partial f/\partial u_{ty} + \cdots.\end{aligned}$$

After collecting all terms that can be written as total derivatives, the symmetry condition (2.1) becomes

$$D_t(M) + D_x(N) + D_y(L) + D_z(K) + E_u(F)\varphi = 0, \tag{2.2}$$

where $E_u(F)$ denotes the Euler–Lagrange operator [18] applied to F,

$$\begin{aligned}E_u(F) = D_t^2(F_{u_{tt}}) + D_x^2(F_{u_{xx}}) + D_y^2(F_{u_{yy}}) + D_z^2(F_{u_{zz}}) + D_tD_x(F_{u_{tx}}) + D_tD_y(F_{u_{ty}})\\ + D_tD_z(F_{u_{tz}}) + D_xD_y(F_{u_{xy}}) + D_xD_z(F_{u_{xz}}) + D_yD_z(F_{u_{yz}})\\ - D_t(F_{u_t}) - D_x(F_{u_x}) - D_y(F_{u_y}) - D_z(F_{u_z}) + F_u\end{aligned} \tag{2.3}$$

and M, N, L, K are defined respectively by

$$\begin{aligned}
M &= F_{u_{tt}}\varphi_t + \tfrac{1}{2}F_{u_{tx}}\varphi_x + \tfrac{1}{2}F_{u_{ty}}\varphi_y + \tfrac{1}{2}F_{u_{tz}}\varphi_z \\
&\qquad + \left[F_{u_t} - D_t\left(F_{u_{tt}}\right) - \tfrac{1}{2}D_x\left(F_{u_{tx}}\right) - \tfrac{1}{2}D_y\left(F_{u_{ty}}\right) - \tfrac{1}{2}D_z\left(F_{u_{tz}}\right)\right]\varphi, \\
N &= F_{u_{xx}}\varphi_x + \tfrac{1}{2}F_{u_{tx}}\varphi_t + \tfrac{1}{2}F_{u_{xy}}\varphi_y + \tfrac{1}{2}F_{u_{xz}}\varphi_z \\
&\qquad + \left[F_{u_x} - D_x\left(F_{u_{xx}}\right) - \tfrac{1}{2}D_t\left(F_{u_{tx}}\right) - \tfrac{1}{2}D_y\left(F_{u_{xy}}\right) - \tfrac{1}{2}D_z\left(F_{u_{xz}}\right)\right]\varphi, \\
L &= F_{u_{yy}}\varphi_y + \tfrac{1}{2}F_{u_{ty}}\varphi_t + \tfrac{1}{2}F_{u_{xy}}\varphi_x + \tfrac{1}{2}F_{u_{yz}}\varphi_z \\
&\qquad + \left[F_{u_y} - D_y\left(F_{u_{yy}}\right) - \tfrac{1}{2}D_t\left(F_{u_{ty}}\right) - \tfrac{1}{2}D_x\left(F_{u_{xy}}\right) - \tfrac{1}{2}D_z\left(F_{u_{yz}}\right)\right]\varphi, \\
K &= F_{u_{zz}}\varphi_z + \tfrac{1}{2}F_{u_{tz}}\varphi_t + \tfrac{1}{2}F_{u_{xz}}\varphi_x + \tfrac{1}{2}F_{u_{yz}}\varphi_y \\
&\qquad + \left[F_{u_z} - D_z\left(F_{u_{zz}}\right) - \tfrac{1}{2}D_t\left(F_{u_{tz}}\right) - \tfrac{1}{2}D_x\left(F_{u_{xz}}\right) - \tfrac{1}{2}D_y\left(F_{u_{yz}}\right)\right]\varphi.
\end{aligned} \tag{2.4}$$

The determining equation, transformed to the form (2.2), takes the divergence form on solutions of (1.1):

$$D_t(M) + D_x(N) + D_y(L) + D_z(K) = 0, \tag{2.5}$$

if and only if the Euler–Lagrange equation

$$E_u(F) = 0 \tag{2.6}$$

is identically satisfied on solutions of $F = 0$, which is equivalent to the 4-divergence form of equation (1.1) itself [18]:

$$F \equiv D_t(P) + D_x(Q) + D_y(R) + D_z(S) = 0, \tag{2.7}$$

where P, Q, R, S depend on the same set of variables as F in (1.1).

3. Two-dimensional divergence form of the symmetry condition

In order to introduce a unique potential as a consequence of a symmetry condition, we have to convert the four-dimensional divergence on the left-hand side of the symmetry condition (2.5) into a two-dimensional divergence, say, in the variables t and x. To do this, we present L and K as the sum of total derivatives in t and x plus remaining terms which cannot be given in this form:

$$\begin{aligned}
L &= D_t\left(\tfrac{1}{2}F_{u_{ty}}\varphi\right) + D_x\left(\tfrac{1}{2}F_{u_{xy}}\varphi\right) + F_{u_{yy}}\varphi_y + \tfrac{1}{2}F_{u_{yz}}\varphi_z \\
&\qquad + \left[F_{u_y} - D_y\left(F_{u_{yy}}\right) - D_t\left(F_{u_{ty}}\right) - D_x\left(F_{u_{xy}}\right) - \tfrac{1}{2}D_z(F_{u_{zy}})\right]\varphi, \\
K &= D_t\left(\tfrac{1}{2}F_{u_{tz}}\varphi\right) + D_x\left(\tfrac{1}{2}F_{u_{xz}}\varphi\right) + F_{u_{zz}}\varphi_z + \tfrac{1}{2}F_{u_{yz}}\varphi_y \\
&\qquad + \left[F_{u_z} - D_z\left(F_{u_{zz}}\right) - D_t\left(F_{u_{tz}}\right) - D_x\left(F_{u_{xz}}\right) - \tfrac{1}{2}D_y\left(F_{u_{yz}}\right)\right]\varphi.
\end{aligned} \tag{3.1}$$

Using (3.1) in (2.5) together with definitions (2.4) and collecting terms with the total derivatives with respect to t and x, we convert (2.5) to the form

$$\begin{aligned}
&D_t(\bar{M}) - D_x(\bar{N}) + F_{u_{yy}}\varphi_{yy} + F_{u_{yz}}\varphi_{yz} + F_{u_{zz}}\varphi_{zz} \\
&\qquad + \left[F_{u_y} - D_t\left(F_{u_{ty}}\right) - D_x\left(F_{u_{xy}}\right)\right]\varphi_y + \left[F_{u_z} - D_t\left(F_{u_{tz}}\right) - D_x\left(F_{u_{xz}}\right)\right]\varphi_z \\
&\qquad + \left\{D_y\left[F_{u_y} - D_y\left(F_{u_{yy}}\right) - D_t\left(F_{u_{ty}}\right) - D_x\left(F_{u_{xy}}\right)\right]\right. \\
&\qquad \left. + D_z\left[F_{u_z} - D_z\left(F_{u_{zz}}\right) - D_t\left(F_{u_{tz}}\right) - D_x\left(F_{u_{xz}}\right)\right] - D_yD_z\left(F_{u_{yz}}\right)\right\}\varphi = 0,
\end{aligned} \tag{3.2}$$

where $\bar{M}$ and $\bar{N}$ are defined respectively by

$$\begin{aligned}
\bar{M} &= F_{u_{tt}}\varphi_t + \tfrac{1}{2}F_{u_{tx}}\varphi_x + F_{u_{ty}}\varphi_y + F_{u_{tz}}\varphi_z + \left[F_{u_t} - D_t\left(F_{u_{tt}}\right) - \tfrac{1}{2}D_x\left(F_{u_{tx}}\right)\right]\varphi, \\
\bar{N} &= -\left\{F_{u_{xx}}\varphi_x + \tfrac{1}{2}F_{u_{tx}}\varphi_t + F_{u_{xy}}\varphi_y + F_{u_{xz}}\varphi_z + \left[F_{u_x} - D_x\left(F_{u_{xx}}\right) - \tfrac{1}{2}D_t\left(F_{u_{tx}}\right)\right]\varphi\right\}.
\end{aligned} \tag{3.3}$$

It is clear that in order to have the symmetry condition (3.2) to be a two-dimensional divergence in the variables t and x, the coefficients of all the terms not included in the total derivatives D_t and D_x should vanish on solutions of (1.1):

$$F_{u_{yy}} = F_{u_{yz}} = F_{u_{zz}} = 0, \tag{3.4}$$

$$F_{u_y} - D_t\big(F_{u_{ty}}\big) - D_x\big(F_{u_{xy}}\big) = 0, \quad F_{u_z} - D_t\big(F_{u_{tz}}\big) - D_x\big(F_{u_{xz}}\big) = 0 \tag{3.5}$$

whereas, as a consequence of (3.4) and (3.5), the coefficient of φ in (3.2) vanishes identically and the symmetry condition (3.2) becomes

$$D_t(\bar{M}) = D_x(\bar{N}). \tag{3.6}$$

Note that the symmetry condition and therefore all equations (3.2), (3.5) and (3.6) should be satisfied not identically but only on solutions of the original PDE (1.1) and hence they should be (differential) consequences of $F = 0$.

Condition (3.6) is equivalent to the local existence of the potential ψ defined by

$$\psi_t = \hat{N} = \bar{N} + \Lambda_t, \qquad \psi_x = \hat{M} = \bar{M} + \Lambda_x, \qquad \Lambda = \omega\varphi, \tag{3.7}$$

where ω may depend on t, x, y, z, u and the first and second derivatives of u. Here the terms with the derivatives of Λ are added in order to have the most general definition of the potential ψ. Now, the symmetry condition (3.2) can be written as

$$D_t(\hat{M}) = D_x(\hat{N}) \tag{3.8}$$

on solutions of $F = 0$.

4. Existence conditions for partner symmetries

Our second requirement is that the potential ψ should also be a symmetry of the PDE (1.1), i.e. a partner symmetry for the original symmetry φ, so that (3.7) becomes a recursion relation for symmetries. Then the symmetry condition in the two-dimensional divergence form (3.8) with φ replaced by ψ, defined by (3.7), should be satisfied on solutions of equation (1.1),

$$D_t(\tilde{M}) - D_x(\tilde{N}) = \hat{F}, \tag{4.1}$$

where $\tilde{M}$ and $\tilde{N}$ are obtained from $\bar{M}$ and $\bar{N}$ respectively by replacing φ with ψ in (3.3),

$$\begin{aligned} \tilde{M} &= F_{u_{tt}}\psi_t + \tfrac{1}{2}F_{u_{tx}}\psi_x + F_{u_{ty}}\psi_y + F_{u_{tz}}\psi_z + \big[F_{u_t} - D_t\big(F_{u_{tt}}\big) - \tfrac{1}{2}D_x\big(F_{u_{tx}}\big)\big]\psi, \\ \tilde{N} &= -\big\{F_{u_{xx}}\psi_x + \tfrac{1}{2}F_{u_{tx}}\psi_t + F_{u_{xy}}\psi_y + F_{u_{xz}}\psi_z + \big[F_{u_x} - D_x\big(F_{u_{xx}}\big) - \tfrac{1}{2}D_t\big(F_{u_{tx}}\big)\big]\psi\big\}. \end{aligned} \tag{4.2}$$

The term $\hat{F}$ has the form

$$\hat{F} = \mu D_t(F) + \nu D_x(F) + \rho D_y(F) + \lambda D_z(F) + \sigma F \tag{4.3}$$

and it accounts for the fact that equation (4.1) should be satisfied only on solutions of (1.1) (a consequence of proposition 2.10 in [18], similar to formula (2.26) therein).

By using in (4.1) definitions (4.2) of $\tilde{M}$ and $\tilde{N}$ and eliminating ψ_t and ψ_x in compliance with definition (3.7) of ψ, the symmetry condition (4.1) becomes

$$\begin{aligned} &D_t\left\{F_{u_{tt}}[\bar{N} + D_t(\omega\varphi)] + \tfrac{1}{2}F_{u_{tx}}[\bar{M} + D_x(\omega\varphi)]\right\} \\ &\quad + D_x\left\{F_{u_{xx}}[\bar{M} + D_x(\omega\varphi)] + \tfrac{1}{2}F_{u_{tx}}\big[\bar{N} + D_t(\omega\varphi)\big]\right\} \\ &\quad + \big[F_{u_{ty}}D_y + F_{u_{tz}}D_z + F_{u_t} - D_t\big(F_{u_{tt}}\big) - \tfrac{1}{2}D_x\big(F_{u_{tx}}\big)\big][\bar{N} + D_t(\omega\varphi)] \\ &\quad + \big[F_{u_{xy}}D_y + F_{u_{xz}}D_z + F_{u_x} - D_x\big(F_{u_{xx}}\big) - \tfrac{1}{2}D_t\big(F_{u_{tx}}\big)\big][\bar{M} + D_x(\omega\varphi)] \\ &\quad + \big[D_t\big(F_{u_{ty}}\big) + D_x\big(F_{u_{xy}}\big)\big]\psi_y + \big[D_t\big(F_{u_{tz}}\big) + D_x\big(F_{u_{xz}}\big)\big]\psi_z \\ &\quad + \big[D_t\big(F_{u_t}\big) - D_t^2\big(F_{u_t t}\big) + D_x\big(F_{u_x}\big) - D_x^2\big(F_{u_x x}\big) - D_t D_x\big(F_{u_{tx}}\big)\big]\psi = \hat{F}, \end{aligned} \tag{4.4}$$

where $\bar{M}$ and $\bar{N}$ are defined in (3.3). Terms with ψ_y, ψ_z and ψ in (4.4) cannot be balanced by any other terms and therefore they should vanish separately on solutions of (1.1) yielding

$$D_t\big(F_{u_{ty}}\big) + D_x\big(F_{u_{xy}}\big) = \hat{F}^y, \qquad D_t\big(F_{u_{tz}}\big) + D_x\big(F_{u_{xz}}\big) = \hat{F}^z, \tag{4.5}$$

$$D_t\big(F_{u_t}\big) + D_x\big(F_{u_x}\big) - D_t^2\big(F_{u_{tt}}\big) - D_x^2\big(F_{u_{xx}}\big) - D_t D_x\big(F_{u_{tx}}\big) = \hat{F}^0 \tag{4.6}$$

respectively, where the terms $\hat{F}^y$, $\hat{F}^z$ and $\hat{F}^0$ are of the same form as (4.3) but with different coefficients. Equations (4.5) together with (3.5) and equation (4.6) together with (2.6), where (3.5) and (2.6) should be satisfied only on solutions of (1.1), imply

$$F_{u_y} = 0, \qquad F_{u_z} = 0, \qquad F_u = 0. \tag{4.7}$$

In all other terms in (4.4), we have replaced ψ_t and ψ_x by expressions (3.7) and $\bar{M}$, $\bar{N}$ should be further replaced with expressions (3.3). We note that, due to definition (3.7) of the potential ψ and its consequence (3.8) (equivalent to (3.6)), φ satisfies the symmetry condition (2.1), which cancels all the terms proportional to ω in (4.4). It is easy to check that all other terms in (4.4) with second derivatives of φ are canceled identically. The remaining terms are proportional to φ_t, φ_x, φ_y, φ_z and φ, so that these five groups of terms should vanish separately on solutions of the equation $F = 0$ to give the following five equations respectively:

$$\begin{aligned} &D_x\big(F_{u_{tt}}F_{u_{xx}}\big) - \tfrac{1}{4}D_x\big(F_{u_{tx}}^2\big) + F_{u_{xy}}D_y\big(F_{u_{tt}}\big) + F_{u_{xz}}D_z\big(F_{u_{tt}}\big)\\ &\quad - \tfrac{1}{2}\big[F_{u_{ty}}D_y\big(F_{u_{tx}}\big) + F_{u_{tz}}D_z\big(F_{u_{tx}}\big)\big] + 2F_{u_{tt}}D_t(\omega) + F_{u_{tx}}D_x(\omega)\\ &\quad + F_{u_{ty}}D_y(\omega) + F_{u_{tz}}D_z(\omega) = \hat{F}_1, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &-\big\{D_t\big(F_{u_{tt}}F_{u_{xx}}\big) - \tfrac{1}{4}D_t\big(F_{u_{tx}}^2\big) + F_{u_{ty}}D_y\big(F_{u_{xx}}\big) + F_{u_{tz}}D_z\big(F_{u_{xx}}\big)\\ &\quad - \tfrac{1}{2}\big[F_{u_{xy}}D_y\big(F_{u_{tx}}\big) + F_{u_{xz}}D_z\big(F_{u_{tx}}\big)\big] - 2F_{u_{xx}}D_x(\omega) - F_{u_{tx}}D_t(\omega)\\ &\quad - F_{u_{xy}}D_y(\omega) - F_{u_{xz}}D_z(\omega)\big\} = \hat{F}_2, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &D_x\big(F_{u_{ty}}F_{u_{xx}}\big) - \tfrac{1}{2}D_x(F_{u_{tx}}F_{u_{xy}}) - D_t(F_{u_{xy}}F_{u_{tt}}) + \tfrac{1}{2}D_t(F_{u_{tx}}F_{u_{ty}})\\ &\quad + F_{u_{xy}}D_y\big(F_{u_{ty}}\big) + F_{u_{xz}}D_z\big(F_{u_{ty}}\big) - F_{u_{ty}}D_y\big(F_{u_{xy}}\big) - F_{u_{tz}}D_z\big(F_{u_{xy}}\big)\\ &\quad + F_{u_{ty}}D_t(\omega) + F_{u_{xy}}D_x(\omega) = \hat{F}_3, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &D_x(F_{u_{tz}}F_{u_{xx}}) - \tfrac{1}{2}D_x(F_{u_{tx}}F_{u_{xz}}) - D_t(F_{u_{xz}}F_{u_{tt}}) + \tfrac{1}{2}D_t(F_{u_{tx}}F_{u_{tz}})\\ &\quad + F_{u_{xy}}D_y\big(F_{u_{tz}}\big) + F_{u_{xz}}D_z\big(F_{u_{tz}}\big) - F_{u_{ty}}D_y\big(F_{u_{xz}}\big) - F_{u_{tz}}D_z\big(F_{u_{xz}}\big)\\ &\quad + F_{u_{tz}}D_t(\omega) + F_{u_{xz}}D_x(\omega) = \hat{F}_4, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &F_{u_t}B - F_{u_x}A + F_{u_{tt}}D_t(B) - F_{u_{xx}}D_x(A) + \tfrac{1}{2}F_{u_{tx}}\big[D_x(B) - D_t(A)\big]\\ &\quad + F_{u_{ty}}D_y(B) + F_{u_{tz}}D_z(B) - F_{u_{xy}}D_y(A) - F_{u_{xz}}D_z(A) + \hat{A}(\omega) = \hat{F}_0, \end{aligned} \tag{4.12}$$

where

$$A = D_t\big(F_{u_{tt}}\big) + \tfrac{1}{2}D_x\big(F_{u_{tx}}\big) - F_{u_t}, \qquad B = D_x\big(F_{u_{xx}}\big) + \tfrac{1}{2}D_t\big(F_{u_{tx}}\big) - F_{u_x} \tag{4.13}$$

and $\hat{A}$ is the operator of the symmetry condition (2.1). Here the terms $\hat{F}_i$, of the form (4.3) but with different coefficients, account for the fact that the equations should be satisfied only on solutions of (1.1). The derivation of equations (4.8), (4.9), (4.10), (4.11) and (4.12) from (4.4) is lengthy but straightforward.

We note that in the notation (4.13) equation (4.6) simplifies to

$$D_t(A) + D_x(B) = \hat{F}. \tag{4.14}$$

5. Equations that admit partner symmetries and recursion relation for symmetries

We proceed now to solve the existence conditions for partner symmetries (4.5), (4.8), (4.9), (4.10), (4.11), (4.12) and (4.14) for the unknown left-hand side F of equation (1.1) and ω in definition (3.7) of the potential ψ. We split these equations with respect to third derivatives of u and obtain over-determined sets of equations which can be easily solved. Our strategy is to choose the function ω and the coefficients $\mu, \nu, \rho, \lambda, \sigma$ in the terms $\hat{F}$ of the form (4.3) in such a way as to have minimum restrictions on the form F of equation (1.1).

We start with equations (4.5), since they do not contain ω. Our strategy results in vanishing of μ, ν, ρ, λ and σ in $\hat{F}^y$ and $\hat{F}^z$ that implies the linear dependence of F on u_{ty}, u_{xy}, u_{tz} and u_{xz}, so that the solution of equations (4.5) has the form

$$\begin{aligned} F = a_1(y,z)(u_{ty}u_{xz} - u_{tz}u_{xy}) + a_2(u_{tx}u_{ty} - u_{tt}u_{xy}) + a_3(u_{ty}u_{xx} - u_{tx}u_{xy}) \\ + a_4(u_{tx}u_{tz} - u_{tt}u_{xz}) + a_5(u_{tz}u_{xx} - u_{tx}u_{xz}) + b_1u_{xy} + b_2u_{ty} \\ + b_3u_{xz} + b_4u_{tz} + g_3(u_{tt}, u_{tx}, u_{xx}, u_t, u_x, t, x, y, z), \end{aligned} \tag{5.1}$$

where the coefficients $a_2, a_3, a_4, a_5, b_1, b_2, b_3$ and b_4 are functions of u_t, u_x, t, x, y, z that satisfy the following equations:

$$\begin{aligned} &a_{3u_t} = a_{2u_x}, \quad b_{2u_t} = a_{2x}, \quad b_{1u_x} = -a_{3t}, \quad b_{2t} = -b_{1x}, \quad a_{3x} = b_{1u_t} + b_{2u_x} + a_{2t}, \\ &a_{5u_t} = a_{4u_x}, \quad b_{4u_t} = a_{4x}, \quad b_{3u_x} = -a_{5t}, \quad b_{4t} = -b_{3x}, \quad a_{5x} = b_{3u_t} + b_{4u_x} + a_{4t}. \end{aligned} \tag{5.2}$$

To simplify the analysis, from now on we assume that all the coefficients in (5.1) are constants, so that all equations (5.2) are identically satisfied, and that g_3 depends only on the second derivatives u_{tt}, u_{tx}, u_{xx}. As a consequence, the left-hand side F of our equation (1.1) depends only on second derivatives of u, so that it takes the form (1.2).

With these restrictions, we substitute expression (5.1) for F in the remaining six equations (4.8), (4.9), (4.10), (4.11), (4.12) and (4.14). The resulting equations are split in third derivatives of u into over-determined sets of equations, where we choose the function ω and the coefficients $\mu, \nu, \rho, \lambda, \sigma$ in the terms of the form (4.3) in such a way as to obtain minimum restrictions on the form of F. It turns out that all these six equations determine only the form of the function g_3:

$$g_3(u_{tt}, u_{tx}, u_{xx}) = a_6\big(u_{tt}u_{xx} - u_{tx}^2\big) + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_0, \tag{5.3}$$

so that equation (1.1) becomes

$$\begin{aligned} F = a_1(u_{ty}u_{xz} - u_{tz}u_{xy}) + a_2(u_{tx}u_{ty} - u_{tt}u_{xy}) + a_3(u_{ty}u_{xx} - u_{tx}u_{xy}) \\ + a_4(u_{tx}u_{tz} - u_{tt}u_{xz}) + a_5(u_{tz}u_{xx} - u_{tx}u_{xz}) + a_6\big(u_{tt}u_{xx} - u_{tx}^2\big) \\ + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_0 = 0 \end{aligned} \tag{5.4}$$

together with the following solution for ω:

$$\omega = -\tfrac{1}{2}(a_2u_{ty} + a_3u_{xy} + a_4u_{tz} + a_5u_{xz}) + \omega_0, \tag{5.5}$$

where all the coefficients in (5.4) and (5.5) are now constants. Using (5.5) in equations (3.7), that define the symmetry potential ψ in terms of the symmetry φ, we obtain the recursion relation between partner symmetries of equation (5.4):

$$\begin{aligned} \psi_t = -(a_2u_{ty} + a_4u_{tz} - a_6u_{tx} + b_6 - \omega_0)\varphi_t - (a_3u_{ty} + a_5u_{tz} + a_6u_{tt} + b_7)\varphi_x \\ + (a_1u_{tz} + a_2u_{tt} + a_3u_{tx} - b_1)\varphi_y + (-a_1u_{ty} + a_4u_{tt} + a_5u_{tx} - b_3)\varphi_z, \\ \psi_x = -(a_2u_{xy} + a_4u_{xz} - a_6u_{xx} - b_5)\varphi_t - (a_3u_{xy} + a_5u_{xz} + a_6u_{tx} - b_6 - \omega_0)\varphi_x \\ + (a_1u_{xz} + a_2u_{tx} + a_3u_{xx} + b_2)\varphi_y + (-a_1u_{xy} + a_4u_{tx} + a_5u_{xx} + b_4)\varphi_z. \end{aligned} \tag{5.6}$$

Indeed, by construction, both φ and ψ satisfy the symmetry condition (2.1) in the divergence form (3.8) and (4.1) respectively, on solutions of (1.2), and hence the transformation (5.6) maps any symmetry φ of equation (5.4) again into its symmetry ψ.

6. Canonical forms of PDEs that admit partner symmetries

Due to the random choice of original variables, both the forms of equation (5.4), that admits partner symmetries, and recursion relation (5.6) contain false generality. Therefore, we consider here the changes of variables that transform equation (5.4) to simple canonical forms and simplify the corresponding recursion relation. This will also make up for our casual choice of variables t and x for the two-dimensional divergence form.

Case I: $a_1 \neq 0$. In this case we can make $a_1 = 1$ by dividing (5.4) over a_1. Transform equation (5.4) to the new variables

$$\tilde{t} = t + a_4 y - a_2 z, \qquad \tilde{x} = x + a_5 y - a_3 z, \qquad \tilde{y} = y, \qquad \tilde{z} = z, \tag{6.1}$$

skip *tilde* in the result and introduce the following notation:

$$\begin{aligned} &\Gamma = a_3 a_4 - a_2 a_5 + a_6, \qquad A = a_4 b_2 - a_2 b_4 + b_5, \qquad B = a_5 b_1 - a_3 b_3 + b_7, \\ &C = a_4 b_1 + a_5 b_2 - a_2 b_3 - a_3 b_4 + 2b_6, \qquad D = b_1 b_4 - b_2 b_3 + b_0. \end{aligned} \tag{6.2}$$

The transformed equation (5.4) becomes

$$\begin{aligned} F = u_{ty} u_{xz} - u_{tz} u_{xy} + \Gamma\left(u_{tt} u_{xx} - u_{tx}^2\right) + A u_{tt} + B u_{xx} + C u_{tx} \\ + b_1 u_{xy} + b_2 u_{ty} + b_3 u_{xz} + b_4 u_{tz} + b_0 = 0. \end{aligned} \tag{6.3}$$

Consider here the following subcases.

Subcase Ia: $\Gamma = 0$. Then, with the change of the unknown

$$u = v + b_1 t z + b_4 x y - b_2 x z - b_3 t y, \tag{6.4}$$

equation (6.3) becomes

$$v_{ty} v_{xz} - v_{tz} v_{xy} + A v_{tt} + B v_{xx} + C v_{tx} + D = 0. \tag{6.5}$$

Subcase Ia1: $A = B = C = D = 0$. Then equation (6.5) reduces to the homogeneous version of the first heavenly equation of Plebañski:

$$u_{ty} u_{xz} - u_{tz} u_{xy} = 0, \tag{6.6}$$

where we have restored the original notation u for the unknown.

Subcase Ia2: $A = B = C = 0$, $D \neq 0$. Then we can set $D = -1$, and equation (6.5) becomes the first heavenly equation of Plebañski:

$$u_{ty} u_{xz} - u_{tz} u_{xy} = 1, \tag{6.7}$$

where we have again replaced v by u. In cases Ia1 and Ia2, the recursion relation for symmetries (5.6) becomes

$$\psi_t = \omega_0 \varphi_t + u_{tz} \varphi_y - u_{ty} \varphi_z, \qquad \psi_x = \omega_0 \varphi_x + u_{xz} \varphi_y - u_{xy} \varphi_z. \tag{6.8}$$

Subcase Ia3: $(A, B, C) \neq (0, 0, 0)$. Then we can always make $D \neq 0$ by substituting

$$v = \tilde{v} + k t^2 + l x^2 + m x t \tag{6.9}$$

into (6.5). Applying the Legendre transformation

$$t = w_p, \qquad x = w_q, \qquad v = w - pw_p - qw_q, \qquad p = -v_t, \qquad q = -v_x \tag{6.10}$$

to equation (6.5), we obtain

$$w_{py}w_{qz} - w_{pz}w_{qy} + Aw_{qq} + Bw_{pp} - Cw_{pq} - D\big(w_{pp}w_{qq} - w_{pq}^2\big) = 0, \tag{6.11}$$

where the new unknown $w = w(p, q, y, z)$ is the Legendre transform of v. Finally, by the change of the unknown

$$w = u + \frac{1}{2D}\,(Ap^2 + Bq^2 + Cpq), \tag{6.12}$$

equation (6.11) takes the form

$$u_{py}u_{qz} - u_{pz}u_{qy} - D\big(u_{pp}u_{qq} - u_{pq}^2\big) + \frac{1}{4D}(4AB - C^2) = 0. \tag{6.13}$$

Since $D \neq 0$, we can always make $D = 1$, by the choice of the constants k, l and m in the transformation (6.9), and the nonvanishing constant term to be ± 1, by an appropriate scaling of u, so that equation (6.13) takes the canonical form

$$u_{ty}u_{xz} - u_{tz}u_{xy} + u_{tt}u_{xx} - u_{tx}^2 = \varepsilon, \tag{6.14}$$

where $\varepsilon = \pm 1$ and p, q are replaced by t, x respectively. We call (6.14) the *mixed heavenly equation*. It is interesting to note that equation (6.14) is form-invariant under the Legendre transformation (6.10) for $\varepsilon = 1$ and under the transformation (6.10), combined with the reflection in t or x, for $\varepsilon = -1$. The homogeneous version of the mixed heavenly equation (6.14), with $\varepsilon = 0$, is transformed to the first heavenly equation under the Legendre transformation (6.10). The recursion relation (5.6) for symmetries of equation (6.14) becomes

$$\begin{aligned} \psi_t &= (u_{tx} + \omega_0)\varphi_t - u_{tt}\varphi_x + u_{tz}\varphi_y - u_{ty}\varphi_z,\\ \psi_x &= u_{xx}\varphi_t - (u_{tx} - \omega_0)\varphi_x + u_{xz}\varphi_y - u_{xy}\varphi_z. \end{aligned} \tag{6.15}$$

Recently we became aware of the relation of the mixed heavenly equation to Husain's heavenly equation (at $\varepsilon = +1$) [13, 14] related to the chiral model approach to self-dual gravity:

$$v_{tz}v_{py} - v_{ty}v_{pz} + v_{tt} + \varepsilon v_{pp} = 0. \tag{6.16}$$

Equation (6.16) can be obtained from the mixed heavenly equation by the partial Legendre transformation in x:

$$x = v_p, \qquad u = v - pv_p, \qquad p = -u_x, \qquad v(t, p, y, z) = u - xu_x. \tag{6.17}$$

We note that (6.16) could also be obtained as a canonical equation in subcase Ia3 of the general equation (5.4) with the replacement $u \mapsto v$, $x \mapsto p$ with the following choice of the coefficients in (5.4): $a_1 = -1$, $b_5 = 1$, $b_7 = \varepsilon$ and all other coefficients vanishing. Then from (5.6) with the same change of notation we obtain the recursion between partner symmetries for equation (6.16):

$$\psi_t = v_{ty}\varphi_z - v_{tz}\varphi_y - \varepsilon\varphi_p + \omega_0\varphi_t, \qquad \psi_p = v_{py}\varphi_z - v_{pz}\varphi_y + \varphi_t + \omega_0\varphi_p. \tag{6.18}$$

Subcase Ib: $\Gamma \neq 0$. In this case, after applying the transformation (6.4), equation (6.3) takes the form (6.11) for the unknown v plus the constant term $D = b_1b_4 - b_2b_3 + b_0$:

$$v_{ty}v_{xz} - v_{tz}v_{xy} + \Gamma\big(v_{tt}v_{xx} - v_{tx}^2\big) + Av_{tt} + Bv_{xx} + Cv_{tx} + D = 0. \tag{6.19}$$

Then the transformation similar to (6.12):

$$v = u - \frac{1}{2\Gamma}\,(Ax^2 + Bt^2 - Ctx), \tag{6.20}$$

being applied to (6.19), results in the same equation (6.14), after setting $\Gamma = 1$ by an appropriate scaling of t and y or x and z.

Case II: $a_1 = 0$.

Subcase IIa: $a_2 = a_3 = a_4 = a_5 = 0$, $a_6 \neq 0$. Then we can set $a_6 = 1$ and equation (5.4) becomes

$$u_{tt}u_{xx} - u_{tx}^2 + b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_0 = 0. \tag{6.21}$$

The substitution

$$u = v - \tfrac{1}{2}(b_5x^2 + b_7t^2) + b_6tx \tag{6.22}$$

transforms equation (6.21) to the form

$$v_{tt}v_{xx} - v_{tx}^2 + b_1v_{xy} + b_2v_{ty} + b_3v_{xz} + b_4v_{tz} + b_0 + b_6^2 - b_5b_7 = 0. \tag{6.23}$$

We assume that $(b_1, b_2, b_3, b_4) \neq (0, 0, 0, 0)$ since otherwise equation (6.23) determines v that depends only on two variables. Let, say, $b_1 \neq 0$. Then, under the substitution

$$v = w - \frac{\left(b_6^2 - b_5b_7 + b_0\right)}{b_1}xy, \tag{6.24}$$

equation (6.23) becomes

$$w_{tt}w_{xx} - w_{tx}^2 + b_1w_{xy} + b_2w_{ty} + b_3w_{xz} + b_4w_{tz} = 0. \tag{6.25}$$

We consider only the case when $(b_1, b_2) \neq (0, 0)$ and $(b_3, b_4) \neq (0, 0)$, since otherwise equation (6.25) would determine a function of less than four variables.

Subcase IIa1: $b_1b_4 - b_2b_3 \neq 0$. Then by the change of variables

$$y' = ay + bz, \qquad z' = cy + dz \tag{6.26}$$

with the appropriate choice of a, b, c, d, equation (6.25) takes the form of the second heavenly equation of Plebañski:

$$u_{tt}u_{xx} - u_{tx}^2 + u_{xy} + u_{tz} = 0, \tag{6.27}$$

where we have again denoted the unknown by u. Note that in case IIa1 the transformation (6.26) has the nonvanishing determinant $ad - bc \neq 0$. The recursion (5.6) for symmetries of (6.27) takes the form

$$\psi_t = (u_{tx} + \omega_0)\varphi_t - u_{tt}\varphi_x - \varphi_y, \qquad \psi_x = u_{xx}\varphi_t - (u_{tx} - \omega_0)\varphi_x + \varphi_z, \tag{6.28}$$

which at $\omega_0 = 0$ coincides, up to the change $\psi \mapsto -\psi$, with our previous result [6].

Subcase IIa2: $b_1b_4 - b_2b_3 = 0$. Then, by choosing a certain linear combination of y and z, we obtain the equation which determines a function of only three variables.

Subcase IIb: $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$. Then (5.4) reduces to the linear equation

$$b_5u_{tt} + 2b_6u_{tx} + b_7u_{xx} + b_1u_{xy} + b_2u_{ty} + b_3u_{xz} + b_4u_{tz} + b_0 = 0. \tag{6.29}$$

The recursion (5.6) for symmetries of (6.29) becomes

$$\psi_t = -(b_6 - \omega_0)\varphi_t - b_7\varphi_x - b_1\varphi_y - b_3\varphi_z, \quad \psi_x = b_5\varphi_t + (b_6 + \omega_0)\varphi_x + b_2\varphi_y + b_4\varphi_z. \tag{6.30}$$

Subcase IIc: $a_1 = 0$, $(a_2, a_3, a_4, a_5) \neq (0, 0, 0, 0)$. Equation (5.4) with $a_1 = 0$ by an appropriate linear transformation of independent variables together with a shift of the unknown by a quadratic form of independent variables, followed by the Legendre transformation in t

and x, in the non-degenerate case, when the equation determines a function of four variables, takes the canonical form

$$u_{tx}u_{ty} - u_{tt}u_{xy} + au_{tz} + bu_{xz} + cu_{xx} = 0, \tag{6.31}$$

up to a possible change of notation for independent variables $x \mapsto t$ and/or $z \mapsto y$. We call this equation *asymmetric heavenly equation*. At $b = 0$ it becomes the so-called *evolution form of the second heavenly equation* [15, 19, 20].

The recursion relation (5.6) for symmetries of (6.31) takes the form

$$\psi_t = -(u_{ty} - \omega_0)\varphi_t - c\varphi_x + u_{tt}\varphi_y - b\varphi_z, \quad \psi_x = -u_{xy}\varphi_t + \omega_0\varphi_x + u_{tx}\varphi_y + a\varphi_z. \tag{6.32}$$

7. Point and contact symmetries of canonical equations

In the method of partner symmetries, we consider a nonlocal symmetry with the characteristic $\hat{\eta} = \tilde{\varphi} - R\varphi$, where $\tilde{\varphi}$ is any point symmetry of our equation and R is the recursion operator determined by recursion relations (5.6), generating a nonlocal symmetry $\psi = R\varphi$ from a point symmetry φ. We search for solutions invariant with respect to a nonlocal symmetry $\hat{\eta}$, determined by the condition $\tilde{\varphi} - R\varphi = 0$ [3], so that we can obtain this invariance condition by formally replacing ψ by a point symmetry $\tilde{\varphi} : \psi = \tilde{\varphi}$ in the recursion relations (5.6). This does not mean symmetry reduction, so that generically these solutions depend on all four variables and so they are still noninvariant solutions in the usual sense. Contact symmetries can also be used for φ and/or $\tilde{\varphi}$. Therefore, in this section we present all point and contact symmetries of the canonical equations, since any of them can be chosen for $\tilde{\varphi}$ and serve as an input for φ into (5.6). All the point and contact symmetries were computed by using the programs 'LIEPDE' and 'CRACK' by Wolf [21] run in the computer algebra system 'REDUCE 3.8'.

The homogeneous version (6.6) of the first heavenly equation of Plebański, in case Ia1, has point symmetries with the following set of generators:

$$\begin{aligned} &X_1 = a(y,z)\partial_y, \qquad X_2 = b(y,z)\partial_z, \qquad X_3 = c(t,x)\partial_t, \qquad X_4 = d(t,x)\partial_x, \\ &X_5 = f(y,z)\partial_u \qquad X_6 = g(t,x)\partial_u, \qquad X_7 = u\partial_u, \end{aligned} \tag{7.1}$$

where a, b, c, d, f and g are arbitrary functions of two specified variables. From now on, all the functions, involved in symmetry generators, are either arbitrary or satisfy certain linear equations.

Nontrivial contact symmetries of (6.6), that are not point symmetries, have the following generating functions [12]:

$$W_1 = a(t, x, u_t, u_x), \qquad W_2 = b(y, z, u_y, u_z) \tag{7.2}$$

that coincide with the symmetry characteristics.

The first heavenly equation of Plebański (6.7), in case Ia2, has only point symmetries with the following set of generators:

$$\begin{aligned} &X_1 = a(y,z)\partial_u, \qquad X_2 = b(t,x)\partial_u, \qquad X_3 = c_y(y,z)\partial_z - c_z(y,z)\partial_y, \\ &X_4 = d_t(t,x)\partial_x - d_x(t,x)\partial_t, \qquad X_5 = 2x\partial_x + u\partial_u, \qquad X_6 = z\partial_z - x\partial_x. \end{aligned} \tag{7.3}$$

The mixed heavenly equation (6.14), in case Ia3, has point symmetries with the following set of generators:

$$\begin{aligned} &X_1 = t\partial_u, \qquad X_2 = t\partial_t - x\partial_x, \qquad X_3 = t\partial_t + z\partial_z + u\partial_u, \qquad X_4 = \partial_x, \\ &X_5 = t\partial_x, \qquad X_6 = \partial_t, \qquad X_7 = x\partial_t, \qquad X_8 = x\partial_u, \qquad X_9 = a(y,z)\partial_u, \\ &X_{10} = b_y(y,z)\partial_z - b_z(y,z)\partial_y, \end{aligned} \tag{7.4}$$

the same for $\varepsilon = \pm 1$.

Nontrivial contact symmetries of (6.14) are determined by the generating function

$$W = f(u_t, u_x, t, x), \tag{7.5}$$

where the function f is an arbitrary solution of the linear equations

$$f_{u_t u_t} + \varepsilon f_{xx} = 0, \qquad f_{u_x u_x} + \varepsilon f_{tt} = 0, \qquad f_{u_t u_x} = \varepsilon f_{tx}, \qquad f_{u_t t} + f_{u_x x} = 0. \tag{7.6}$$

The homogeneous version of the mixed heavenly equation

$$u_{ty}u_{xz} - u_{tz}u_{xy} + u_{tt}u_{xx} - u_{tx}^2 = 0 \tag{7.7}$$

has point symmetries with the following generators:

$$\begin{aligned} &X_1 = a(y,z)\partial_u, \qquad X_2 = b_y(y,z)\partial_z - b_z(y,z)\partial_y, \qquad X_3 = x\partial_u, \\ &X_4 = x\partial_t, \qquad X_5 = t\partial_t + z\partial_z, \qquad X_6 = \partial_t, \qquad X_7 = t\partial_x, \qquad X_8 = \partial_x, \\ &X_9 = u\partial_u, \qquad X_{10} = x\partial_x + z\partial_z, \qquad X_{11} = t\partial_u. \end{aligned} \tag{7.8}$$

Nontrivial contact symmetries of (7.7) have the generating functions

$$W_1 = x f_{u_t}(u_t, u_x) - t f_{u_x}(u_t, u_x), \qquad W_2 = g(u_t, u_x), \tag{7.9}$$

where f and g are arbitrary functions of u_t, u_x.

Lie point symmetries of (6.16), which at $\varepsilon = +1$ is Husain's equation, have the following generators:

$$\begin{aligned} &X_1 = y\partial_y + v\partial_v, \qquad X_2 = a_z(y,z)\partial_y - a_y(y,z)\partial_z, \qquad X_3 = b(y,z)\partial_v, \\ &X_4 = c_t(t,p)\partial_t - \varepsilon c_p(t,p)\partial_p + d(t,p)\partial_v, \end{aligned} \tag{7.10}$$

where $\varepsilon = \pm 1$ is the coefficient of the term v_{pp} in (6.16) and $c(t,p)$ and $d(t,p)$ should satisfy the equations

$$c_{tt}(t,p) + \varepsilon c_{pp}(t,p) = 0, \qquad d_{tt}(t,p) + \varepsilon d_{pp}(t,p) = 0.$$

Nontrivial Lie contact symmetries of (6.16) are determined by the generating function $W = W(t, p, v_t, v_p)$ which should satisfy the equations

$$W_{v_t t} = W_{v_p p}, \quad W_{v_t p} + \varepsilon W_{v_p t} = 0, \quad W_{v_t v_t} + \varepsilon W_{v_p v_p} = 0, \quad W_{tt} + \varepsilon W_{pp} = 0. \tag{7.11}$$

The second heavenly equation (6.27), in case IIa1, has only point symmetries with the set of generators

$$\begin{aligned} &X_1 = a(y,z)\partial_u, \qquad X_2 = (xb_z(y,z) - tb_y(y,z))\partial_u, \qquad X_3 = t\partial_t + y\partial_y + u\partial_u, \\ &X_4 = x\partial_x - y\partial_y + 2u\partial_u, \qquad X_5 = 2y\partial_t - tx\partial_u, \\ &X_6 = (tc_{yz}(y,z) - xc_{zz}(y,z))\partial_t + (tc_{yy}(y,z) - xc_{yz}(y,z))\partial_x + c_z(y,z)\partial_y \\ &\quad - c_y(y,z)\partial_z + \tfrac{1}{2}\left(tx^2c_{yzz}(y,z) + \tfrac{1}{3}t^3c_{yyy}(y,z) - t^2xc_{yyz}(y,z) - \tfrac{1}{3}x^3c_{zzz}(y,z)\right)\partial_u, \\ &X_7 = d_z(y,z)\partial_t + d_y(y,z)\partial_x + \left(\tfrac{1}{2}t^2d_{yy}(y,z) - txd_{yz}(y,z) + \tfrac{1}{2}x^2d_{zz}(y,z)\right)\partial_u. \end{aligned} \tag{7.12}$$

These symmetries had been already presented in our paper [6] together with the commutator table and the same symmetries were also obtained later in the paper [22] by using a different computer program 'LIE'. We reproduce these symmetries here to make the paper self-contained.

We skip case IIb of the linear equation (6.29), since we concentrate here on nonlinear equations, so we proceed to the asymmetric heavenly equation (6.31) in case IIc. The point symmetries of (6.31) have the following generators:

$$X_1 = A(y,z)\partial_u, \qquad X_2 = y\partial_y + u\partial_u, \qquad X_3 = B_y(y,z)\partial_t + (bt - ax)B_z(y,z)\partial_u \tag{7.13}$$

plus one more lengthy generator

$$X_4 = (tf''(z) + bkt - h_x(x,z))\partial_t + \left(\frac{c}{b}f'(z) + k\theta\right)\partial_x + (g'(z) - yf''(z) - 2bky)\partial_y + f'(z)\partial_z \\ + \left\{\left(\frac{b}{2}t^2 - atx\right)(yf'''(z) - g''(z)) + ayh_z(x,z) + d(x,z) - 2ackty\right\}\partial_u, \qquad (7.14)$$

where $f(z)$ and $g(z)$ are arbitrary functions, $\theta = bx - cz$ and the functions $d(x, z)$ and $h(x, z)$ are defined as follows:

$$d(x,z) = \Phi(\theta) - \frac{a^2}{2b^2}[b^2x^2g''(z) - 2bcxg'(z) + 2c^2g(z)], \qquad (7.15)$$

$$h(x,z) = \Phi(\theta) + \frac{ack}{2b^2}(2z\theta + cz^2) + \frac{a}{2b^3}[b^2x^2f''(z) - 2bcxf'(z) + 2c^2f(z)] \qquad (7.16)$$

and $\Phi(\theta)$ is an arbitrary function.

Contact symmetries of (6.31) are determined by the following generating function:

$$W = \Phi(\alpha,\theta) + \left[\frac{\beta}{b}g_{yz}(y,z) - \frac{h_y(y,z)}{b} - \frac{(k-l)}{b^2}(acz + b^2t)\right]u_t + \frac{g_{yy}(y,z)}{2b}u_t^2 \\ - \frac{(k-l)\theta}{b}u_x + (g_z(y,z) - 2ly)u_y - g_y(y,z)u_z + 2ku + \frac{\beta^2}{2b}g_{zz}(y,z) - \frac{c\theta}{b^2}g_y(y,z) \\ - \frac{a^2c}{b^2}xg_z(y,z) + \frac{a^2c^2}{b^3}g - \frac{\beta}{b}h_z(y,z) + \frac{ac}{b^2}h - \frac{a^2c}{b^3}(k-l)y\theta. \qquad (7.17)$$

For symmetry generators of point transformations in the geometric form

$$X = \xi^t\partial_t + \xi^x\partial_x + \xi^y\partial_y + \xi^z\partial_z + \eta\partial_u \qquad (7.18)$$

the symmetry characteristic [18] is defined as

$$\hat{\eta} = \eta - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z. \qquad (7.19)$$

For each of the canonical equations, characteristics $\hat{\eta}$ of point symmetries or generating functions W of contact symmetries can be used for φ and ψ in the recursion relations, as was explained at the beginning of this section. Then the recursions become differential constraints that determine particular, generically noninvariant, solutions of these equations.

8. Lift from invariant to noninvariant solutions of the mixed heavenly equation

Here we demonstrate the application of partner symmetries for obtaining noninvariant solutions of canonical PDEs and, in particular, a *lift from invariant to noninvariant solutions.* We choose mixed heavenly equation (6.14) as an example, possessing the recursion for symmetries (6.15), where we set $\omega_0 = 0$. Equation (6.14) admits the obvious translational symmetry with the generator $X = \partial_x + \partial_z$.

Solutions, *invariant* under this symmetry, have the form $u = u(s, t, y)$, where $s = x - z$, since they do not change under the simultaneous shift in x and z. Then u satisfies the reduced equation

$$u_{ts}u_{sy} - u_{ty}u_{ss} + u_{tt}u_{ss} - u_{ts}^2 = \varepsilon, \qquad (8.1)$$

obtained from (6.14) by the symmetry reduction. Under the Legendre transformation

$$r = u_s, \qquad v(r,t,y) = u - su_s, \qquad s = -v_r, \qquad u = v - rv_r \qquad (8.2)$$

equation (8.1) is linearized in the form

$$v_{tt} + \varepsilon v_{rr} - v_{ty} = 0. \tag{8.3}$$

By using partner symmetries, we shall show that solutions of the linear equation (8.3), i.e. *invariant solutions* of Legendre transformed mixed heavenly equation, being written in certain new coordinates, *can be lifted up to noninvariant solutions* of the latter equation.

As was explained at the beginning of section 7, we formally replace ψ, that is generated from a point symmetry φ in the recursion relations (6.15) (with $\omega_0 = 0$), by a point symmetry $\tilde{\varphi}$. Here we choose both φ and $\psi = \tilde{\varphi}$ to be the indicated above combination of translations in x and z with the characteristic $\psi = \varphi = u_x + u_z$, so that (6.15) becomes

$$u_{xx} + u_{xz} = u_{tz}u_{xx} - u_{tx}u_{xz} + u_{xz}u_{yz} - u_{xy}u_{zz}, \tag{8.4}$$

$$u_{tx} + u_{tz} = u_{tx}(u_{tx} + u_{tz}) - u_{tt}(u_{xx} + u_{xz}) + u_{tz}(u_{xy} + u_{yz}) - u_{ty}(u_{xz} + u_{zz}). \tag{8.5}$$

With the aid of (6.14), equation (8.5) takes the form

$$u_{tx} + u_{tz} = u_{tx}u_{tz} - u_{tt}u_{xz} + u_{tz}u_{yz} - u_{ty}u_{zz} - \varepsilon. \tag{8.6}$$

After the Legendre transformation

$$p = u_x, \quad q = u_z, \quad v(p, q, t, y) = u - xu_x - zu_z, \quad x = -v_p, \quad z = -v_q \tag{8.7}$$

equations (8.4) and (8.6) take the form

$$v_{pq} = v_{qq} + v_{tq} - v_{py}, \tag{8.8}$$

$$v_{pq}(v_{tq} - \varepsilon v_{pq} + v_{tt}) = v_{pp}(-\varepsilon v_{qq} + v_{tq} + v_{ty}), \tag{8.9}$$

where equation (8.8) was used in the Legendre transform of (8.6) to arrive at (8.9). Equation (8.9) can be set into a linear form

$$\lambda v_{pq} = v_{tq} - \varepsilon v_{qq} + v_{ty}, \tag{8.10}$$

$$\lambda v_{pp} = v_{tq} - \varepsilon v_{pq} + v_{tt} \tag{8.11}$$

by introducing an extra unknown λ depending on all the variables. Solving algebraically the system of the three linear equations (8.8), (8.10) and (8.11) with respect to the principal derivatives v_{ty}, v_{tq} and v_{py} in terms of the remaining parametric derivatives in the form

$$v_{ty} = \varepsilon(v_{qq} - v_{pq}) + \lambda(v_{pq} - v_{pp}) + v_{tt}, \tag{8.12}$$

$$v_{tq} = \varepsilon v_{pq} + \lambda v_{pp} - v_{tt}, \tag{8.13}$$

$$v_{py} = (\varepsilon - 1)v_{pq} + \lambda v_{pp} + v_{qq} - v_{tt}, \tag{8.14}$$

we easily check that all cross derivatives of the left-hand sides coincide as a consequence of these equations, so that this system of PDEs does not have nontrivial integrability conditions. The mixed heavenly equation (6.14) after the Legendre transformation (8.7) becomes

$$v_{tq}v_{py} - v_{pq}v_{ty} + v_{tt}v_{qq} - v_{tq}^2 + \varepsilon\left(v_{pp}v_{qq} - v_{pq}^2\right) = 0. \tag{8.15}$$

We note that the mixed heavenly equation does not need to be form-invariant under the Legendre transformation (8.7) because it is performed with respect to different variables than the Legendre transformation (6.10). The Legendre transformed mixed heavenly equation (8.15) obviously constitutes another particular case of our general equation (5.4), up to a change of notation of the dependent and independent variables. The linear equations (8.12), (8.13), (8.14) together with (8.15) imply that $\lambda = -\varepsilon$ as far as $v_{pp}v_{qq} - v_{pq}^2 \neq 0$. Then the Legendre transformed mixed heavenly equation (8.15) becomes an algebraic consequence of these three linear PDEs with constant coefficients.

In the case when $\varepsilon = -1$ and hence $\lambda = 1$, equations (8.8) and (8.10) imply

$$v_{ty} + v_{py} = 0, \tag{8.16}$$

which can be integrated to yield the linear first-order equation

$$v_t + v_p = C(t, p, q). \tag{8.17}$$

This obviously leads to the dependence of v on the characteristic combination $t - p$ and thus determines invariant solutions. In this case we have a symmetry reduction and no lift to noninvariant solutions.

In the case when $\varepsilon = 1$ and hence $\lambda = -1$, equations (8.8), (8.10) and (8.11) do not imply any linear first-order consequences, so there is no symmetry reduction of the number of variables in this case and invariant solutions generically do not arise. Under the change of variables $(q, p, t) \mapsto (q, \eta = p + t, \xi = p - t)$, equation (8.13) takes the form of the linear reduced equation (8.3) but written in the new variables η, ξ and q:

$$v_{\eta\eta} + v_{\xi\xi} - v_{\xi q} = 0 \tag{8.18}$$

and containing the fourth argument y of the unknown v as a parameter. Any solution of this linear equation depends on the three variables η, ξ and q with the 'constants' of integration depending on the fourth variable y. Certain appropriate linearly independent combinations of two other equations (8.12) and (8.14), with the use of (8.18), in the new variables take the form

$$v_{\xi q} - v_{\eta q} + v_{\xi y} = 0, \tag{8.19}$$

$$v_{\xi q} + v_{\eta q} - v_{qq} + v_{\eta y} = 0. \tag{8.20}$$

These two equations determine the y-dependence of the 'constants' of integration in the solution of (8.18) and hence we obtain the lift of invariant solutions of the Legendre transformed mixed heavenly equation (8.15) to noninvariant solutions of this equation.

It is easy to obtain an infinite set of exact solutions to linear equations with constant coefficients. Indeed, we can try the exponential dependence of v on η:

$$v = \exp(a\eta + b(y)) f(\xi, q, y),$$

so that (8.18) becomes

$$f_{\xi\xi} - f_{\xi q} + a^2 f = 0. \tag{8.21}$$

For the solution of this equation we can try the following ansatz:

$$f = A\cos(\alpha\xi + \beta q + \theta(y)) + B\sin(\alpha\xi + \beta q + \theta(y)). \tag{8.22}$$

Expression (8.22) satisfies (8.21) only if $a = \pm\alpha\sqrt{\alpha - \beta}$, so that

$$v = \exp(\pm\alpha\sqrt{\alpha - \beta}\,\eta + b(y))[A\cos(\alpha\xi + \beta q + \theta(y)) + B\sin(\alpha\xi + \beta q + \theta(y))]. \tag{8.23}$$

Expression (8.23) satisfies (8.19) if

$$\theta(y) = -\beta y, \qquad b(y) = \pm\beta\sqrt{\frac{\alpha - \beta}{\alpha}}\,y$$

so that (8.23) finally becomes

$$v = \exp\left(\pm\sqrt{\alpha(\alpha - \beta)}\left(\eta + \frac{\beta}{\alpha}y\right)\right) \times \{A\cos[\alpha\xi + \beta(q - y)] + B\sin[\alpha\xi + \beta(q - y)]\}. \tag{8.24}$$

Surprisingly enough, this expression satisfies identically the third equation (8.20), though it is linearly independent of the other two equations.

Any linear combination of the solutions of the form (8.24) is again a solution of the three linear equations (8.18) (8.19) and (8.20) and hence, with $\eta = p + t$, $\xi = p - t$, it will satisfy the nonlinear Legendre transformed mixed heavenly equation (8.15) at $\varepsilon = 1$, since it is a consequence of these linear equations. In the case of a discrete spectrum, we can choose, for example, the following linear combination:

$$v = \sum_i \exp\left(\pm\sqrt{\alpha_i(\alpha_i - \beta_i)}\left(\eta + \frac{\beta_i}{\alpha_i}y\right)\right)\{A_i \cos\left[\alpha_i\xi + \beta_i(q - y)\right] + B_i \sin[\alpha_i\xi + \beta_i(q - y)]\}, \tag{8.25}$$

where α_i, β_i, A_i and B_i stand for arbitrary constants. This is an example of a solution to (8.15), which is obviously noninvariant because it clearly depends on four independent combinations of the variables η, ξ, q and y. For the case of a continuous spectrum, the sum in (8.25) should be replaced by an integral.

There is also a class of polynomial solutions. We start with the ansatz

$$v = A(\eta, \xi, y)\frac{q^2}{2} + B(\eta, \xi, y)q + C(\eta, \xi, y). \tag{8.26}$$

Expression (8.26) will satisfy linear equations (8.18) (8.19) and (8.20) if the coefficients have the form

$$\begin{aligned} A(\eta, \xi, y) &= 3[4g(\eta^2 - \xi^2) + 2h\eta\xi + ky^2], \\ B(\eta, \xi, y) &= 3\{[(4g + h)(\xi^2 - \eta^2) + 2(4g - h)\eta\xi]y + h\eta\xi^2 - 4g\eta^2\xi + \mu(\xi^2 - \eta^2)\}, \\ C(\eta, \xi, y) &= k\eta y^3 + 3[h(\eta^2 - \xi^2) - 8g\eta\xi]y^2 + f(\xi\eta^3 - \eta\xi^3) + (h\eta + \mu)\xi^3 - g\eta^4 \\ &\quad + [h\xi^3 + 8g\eta^3 + 12g\eta^2\xi - 3(4g + h)\eta\xi^2 + 3\mu(\eta^2 - \xi^2) - 6\mu\eta\xi]y, \end{aligned} \tag{8.27}$$

where f, g, h, k and μ are arbitrary constants.

For solutions independent of η we obtain, for example, $v = (\xi + q - y)^4$ and, since all the three equations are linear, the sum of this solution and (8.27) is again a solution, so that

$$v = A(\eta, \xi, y)\frac{q^2}{2} + B(\eta, \xi, y)q + C(\eta, \xi, y) + D(\xi + q - y)^4, \tag{8.28}$$

with A, B and C defined by (8.27) and constant D, will satisfy the nonlinear Legendre transformed mixed heavenly equation (8.15). This solution generically depends on all the four variables and hence it is a noninvariant solution, i.e. it does not admit Lie symmetries. More general polynomial solutions can easily be constructed. The sum of the exponential solution (8.25) and a polynomial solution again satisfies (8.15).

9. Ricci-flat metrics governed by the mixed heavenly equation and its Legendre transform

In section 8, we have obtained noninvariant solutions (8.25) and (8.28) of the Legendre transformed mixed heavenly equation (8.15). In order to get the corresponding solution of the mixed heavenly equation (6.14), we had to perform the Legendre transformation of solutions (8.25) and (8.28), inverse to (8.7), which is quite a difficult problem.

Instead, we shall proceed, as we did before in [3, 6, 4], by taking into account that, similar to the complex Monge–Ampère equation and second heavenly equation of Plebañski, the mixed heavenly equation determines a potential that governs Ricci-flat metrics in the self-dual gravity. If we are interested only in such metrics as our final result, then instead of performing the inverse Legendre transformation of our solution, we make the direct Legendre transformation (8.7) of the metric related to the mixed heavenly equation. Then our solutions (8.25) and (8.28) of the Legendre transformed mixed heavenly equation (8.15) at $\varepsilon = 1$, or any other of its solutions determined by the linear equations (8.19), (8.18) and (8.20), will yield a potential governing the Legendre transformed *mixed heavenly metric*.

In order to obtain Ricci-flat metrics related to the mixed heavenly equation, we start with Husain's heavenly metric and then use the relation between Husain's equation and mixed heavenly equation. Husain's heavenly metric has the form [13]

$$\mathrm{d}s^2 = 2\left(\omega_t\,\mathrm{d}t + \omega_p\,\mathrm{d}p + \frac{\left(\omega_t^2 + \omega_p^2\right)}{\Delta_{tp}}\right), \tag{9.1}$$

where

$$\omega_t = \Lambda_{ty}\,\mathrm{d}y + \Lambda_{tz}\,\mathrm{d}z, \qquad \omega_p = \Lambda_{py}\,\mathrm{d}y + \Lambda_{pz}\,\mathrm{d}z, \qquad \Delta_{tp} = \Lambda_{ty}\Lambda_{pz} - \Lambda_{tz}\Lambda_{py}$$

with $\Lambda(t, p, y, z)$ satisfying Husain's equation

$$\Lambda_{tt} + \Lambda_{pp} + \Lambda_{tz}\Lambda_{py} - \Lambda_{ty}\Lambda_{pz} = 0. \tag{9.2}$$

By the one-dimensional Legendre transformation

$$\Lambda = u - xu_x, \qquad p = -u_x, \qquad x = \Lambda_p, \qquad u = \Lambda - p\Lambda_p, \tag{9.3}$$

where the inverse transformation is also given, Husain's equation (9.2) is mapped into the mixed heavenly equation with $\varepsilon = +1$:

$$u_{ty}u_{xz} - u_{tz}u_{xy} + u_{tt}u_{xx} - u_{tx}^2 = 1 \tag{9.4}$$

for the unknown $u(t, x, y, z)$. Performing the transformation (9.3) of Husain's metric (9.1), we obtain the metric governed by equation (9.4):

$$\mathrm{d}s^2 = 2\left\{\omega_t\,\mathrm{d}t + \omega_x\,\mathrm{d}x + \frac{1}{u_{xx}\Delta}\left[(u_{xx}\omega_t - u_{tx}\omega_x)^2 + (\Delta + 1)\omega_x^2\right]\right\}, \tag{9.5}$$

where

$$\omega_t = u_{ty}\,\mathrm{d}y + u_{tz}\,\mathrm{d}z, \qquad \omega_x = u_{xy}\,\mathrm{d}y + u_{xz}\,\mathrm{d}z, \qquad \Delta = u_{tz}u_{xy} - u_{ty}u_{xz}. \tag{9.6}$$

By using a REDUCE program, it has been checked that the metric (9.5) is Ricci-flat as a consequence of equation (9.4).

The asymmetry of the metric (9.5) in variables t and x is caused by the Legendre transformation (9.3) between p and x, which leaves t untransformed. To amend this lack of symmetry, we symmetrize the metric (9.5) in $t \leftrightarrow x$ and $y \leftrightarrow z$ and then introduce $t \pm x$ and $y \pm z$ as new coordinates, which we call again t, x, y, z. The resulting *mixed heavenly metric* has the form

$$\mathrm{d}s^2 = 2\left\{\omega_t\,\mathrm{d}t + \omega_x\,\mathrm{d}x + \frac{1}{\Delta}\left(u_{xx}\omega_t^2 - 2u_{tx}\omega_t\omega_x + u_{tt}\omega_x^2\right)\right\}. \tag{9.7}$$

This metric is also Ricci-flat, provided that the potential u satisfies the mixed heavenly equation (6.14) *for both signs of* ε. Now we apply the Legendre transformation (8.7) to the mixed heavenly metric (9.7) with the result

$$\begin{aligned}\mathrm{d}s^2 = 2\bigg\{&\frac{1}{\delta}\,(v_{tt} + \varepsilon v_{pp})(v_{qt}\,\mathrm{d}t + v_{qp}\,\mathrm{d}p + v_{qq}\,\mathrm{d}q)^2\\ &+ (v_{qt}\,\mathrm{d}t + v_{qp}\,\mathrm{d}p + v_{qq}\,\mathrm{d}q)\left[-\mathrm{d}q + \frac{2}{\delta}(v_{tq}v_{ty} + \varepsilon v_{pq}v_{py})\,\mathrm{d}y\right]\\ &+ \left[v_{yt}\,\mathrm{d}t + v_{yp}\,\mathrm{d}p + \frac{v_{qq}}{\delta}\left(v_{yt}^2 + \varepsilon v_{yp}^2\right)\mathrm{d}y\right]\mathrm{d}y\bigg\},\end{aligned} \tag{9.8}$$

where $\delta = v_{ty}v_{pq} - v_{tq}v_{py}$ and the metric potential $v(t, p, q, y)$ should satisfy the Legendre transformed mixed heavenly equation (8.15). With the latter condition satisfied, by using REDUCE we have checked that the metric (9.8) is Ricci-flat and calculated the Riemann curvature tensor components for an arbitrary v satisfying (8.15). The expressions for these components are too lengthy to be presented for publication. However, the denominators of the Riemann tensor components are simple, so that possible singularities of the curvature tensor either coincide with the singularities of the metric (9.8), being at $\delta \equiv v_{ty}v_{pq} - v_{tq}v_{py} = 0$, or are located at $v_{qq} = 0$, for v being a linear function of q. For the polynomial solution (8.28) the condition $\delta = 0$ could be satisfied only if all the essential coefficients in (8.28) vanished: $h = g = \mu = D = 0$, which would contradict the noninvariance of this solution. The only singularity of the metric corresponding to (8.28) is located at infinity.

As was shown in section 8, we can use any solution of the three linear equations (8.18) (8.19) and (8.20), which imply (8.15) at $\varepsilon = 1$ as their algebraic consequence. In particular, we can use the noninvariant solutions (8.25) and (8.28) for v in the metric (9.8). For noninvariant solutions, there will be no symmetry reduction, so that v will depend on all the four independent variables, which is a necessary (and often sufficient) condition for the metric components in (9.8) to depend also on all the four independent variables. For the exponential solutions to the complex Monge–Ampère equation and to the second heavenly equation, which are similar to (8.25), we have proved in [6] that the corresponding Kähler metric and the second heavenly metric admit no Killing vectors. Similarly, for the solution (8.25) we also expect that the Legendre transformed mixed heavenly metric (9.8) will admit no Killing vectors and hence no symmetry reduction in the number of independent variables will occur.

10. Conclusion

In the theory of gravitational instantons, heavenly metrics with no Killing vectors (no continuous symmetries) can only be generated by noninvariant solutions of CMA. Therefore, we are faced with the problem of obtaining noninvariant solutions of partial differential equations. Partner symmetries proved to be an appropriate tool for solving such a problem because noninvariant solutions can be obtained as solutions invariant with respect to a certain nonlocal symmetry closely related to partner symmetries. Thus, the existence of partner symmetries for a given PDE is necessary to apply this method. In this paper, we have obtained a general form of the scalar second-order PDE in four variables, containing only second derivatives of the unknown, that possesses partner symmetries. Using point and Legendre transformations, we have transformed this general equation to different simplest canonical forms and so presented a classification of inequivalent equations which admit partner symmetries, together with recursion relations for symmetries. Among these equations we find the well-known first and second heavenly equations of Plebañski and two other nonlinear equations which we have called the *mixed heavenly equation* and *asymmetric heavenly equation*. The mixed heavenly equation is related by a partial Legendre transformation to Husain's heavenly equation arising in the chiral model approach to self-dual gravity. A particular case of the asymmetric heavenly equation is the evolution form of the second heavenly equation.

We ignored here all the cases when the canonical equation explicitly contains only three variables. We leave for the future a classification of PDEs with three variables, that admit partner symmetries.

We have determined all point and contact symmetries of the canonical equations because any such symmetries can be used to generate partner symmetries. As an example of application of partner symmetries, we have shown how to construct noninvariant solutions of the Legendre transformed mixed heavenly equation. By applying Legendre transformation in two variables, the latter equation and differential constraints, that are obtained from recursion relations for partner symmetries, have been transformed to a set of three linear equations with constant coefficients that imply the Legendre transformed mixed heavenly equation as their algebraic consequence. One of these equations involves only three variables and formally coincides with a certain reduced equation, which determines invariant solutions of the Legendre transformed mixed heavenly equation, but written in new variables and containing also the fourth variable as a parameter. Two other equations, involving all the four variables, provide a lift from invariant to noninvariant solutions of the Legendre transformed mixed heavenly equation. We have obtained Ricci-flat metrics governed by the mixed heavenly equation and the Legendre transformed mixed heavenly equation. Using any noninvariant solution of the three linear PDEs, we satisfy the necessary condition of arriving at Ricci-flat metrics with metric components depending on all four independent variables. Such metrics will admit no continuous symmetries and no Killing vectors.

Thus, we conclude that, for a scalar second-order PDE with four independent variables, the existence of partner symmetries happens to be a characteristic feature of the equations that describe self-dual gravity in different variables. The partner symmetries provide a tool for obtaining noninvariant solutions of these equations and Ricci-flat self-dual metrics with no Killing vectors.

Acknowledgments

One of the authors (MBS) is grateful to George Bluman for illuminating discussions. The authors thank the referees for their important remarks, which hopefully resulted in an essential improvement of our paper. The research of MBS is partly supported by the research grant from Boğaziçi University Scientific Research Fund, research project no 07B301.

References

[1] Plebañski J F 1975 *J. Math. Phys.* **16** 2395–402
[2] Eguchi T, Gilkey P H and Hanson A J 1980 *Phys. Rep.* **66** 213–393
[3] Malykh A A, Nutku Y and Sheftel M B 2003 *J. Phys. A: Math. Gen.* **36** 10023–37
[4] Malykh A A, Nutku Y and Sheftel M B 2007 *J. Phys. A: Math. Theor.* **40** 9371–86
[5] Ibragimov N H 1984 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)
[6] Malykh A A, Nutku Y and Sheftel M B 2004 *J. Phys. A: Math. Gen.* **37** 7527–45 (arXiv:math-ph/030503)
[7] Sheftel M B and Malykh A A 2008 *J. Nonlinear Math. Phys.* **15** (Suppl 3) 375–85
[8] Dunajskii M and Mason L J 2000 *Commun. Math. Phys.* **213** 641–72
[9] Dunajskii M and Mason L J 2003 arXiv:math.DG/0301171
[10] Malykh A A and Sheftel M B 2000 *Proc. 2nd Int. Conf. 'Symmetry and Differential Equations' (Krasnoyarsk)* ed V K Andreev and Yu V Shan'ko (Krasnoyarsk: Krasnoyarsk University Press) pp 151–4
[11] Malykh A A and Sheftel M B 2002 *Proc. 3rd Int. Conf. 'Symmetry and Differential Equations' (Krasnoyarsk)* ed V K Andreev and Yu V Shan'ko (Krasnoyarsk: Krasnoyarsk University Press) pp 157–60
[12] Bluman G and Kumei S 1989 *Symmetries and Differential Equations* (New York, Heidelberg, Berlin: Springer)
[13] Husain V 1994 *Class. Quantum Grav.* **11** 927–37 (arXiv:gr-qc/9310003)
[14] Plebañski J F, Przanowski M and García-Compeán H 1996 *Mod. Phys. Lett.* A **11** 663–74 (arXiv: hep-th/9509092v2)
[15] Plebañski J F and Przanowski M 1996 *Phys. Lett.* A **212** 22–8 (arXiv:hep-th/9605233)
[16] Ferapontov E V and Khusnutdinova K R 2003 arxiv:nlin/0312015v1 [nlin.SI]
[17] Ferapontov E V, Hadjikos L and Khusnutdinova K R 2007 arXiv:0705.1774 [math.DG]
[18] Olver Peter J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[19] Boyer C P and Plebañski J F 1977 *J. Math. Phys.* **18** 1022
[20] Finley III J D, Plebañski J F, Przanowski M and García-Compeán H 1993 *Phys. Lett.* A **181** 435
[21] Wolf T 1985 *J. Comput. Phys.* **60** 437–46
[22] Abraham-Shrauner B 2007 *Phys. Lett.* A **369** 299–302